Reddit Macro Course: Math Review

Integral

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1 Recommended Problems

1. Let's warm up with a classic two-good Cobb-Douglas problem. The problem is:

$$\max_{x_1,x_2} \quad \alpha \log x_1 + \beta \log x_2$$
$$\text{s.t.} \quad p_1 x_1 + p_2 x_2 = m$$

   (a) Take first-order conditions with respect to x_1, x_2, and the multiplier.

   (b) Solve the problem. A *solution* is a pair of functions $x_1(p_1, p_2, m)$ and $x_2(p_1, p_2, m)$. We call the vector (p_1, p_2, m) the *state* of the system. We call the vector (α, β) the *parameters* of the system.

   (c) Define the *income share* of good 1 to be p_1x_1/m and of good 2 to be p_2x_2/m. Find an expression for the income shares of good 1 and good 2 that only depends on α and β.

   Why do this problem? Because it's a standard Cobb-Douglas utility function and is used relentlessly throughout economics.

2. Now let's do something more interesting. This is a one-period consumption/leisure problem. The consumer gains utility from consumption c and gains *dis*utility from work h. Unlike the previous question, income is no longer exogenous, but is partially determined by the choice of labor hours. The formal problem is:

$$\max_{c,h} \quad \log c - \frac{\chi}{1+\eta} h^{1+\eta}$$
$$\text{s.t.} \quad c = wh + d$$

   (a) Take first-order conditions with respect to c, h, and the multiplier. The state is (w, d). The parameters are (χ, η).

   (b) Eliminate the multiplier to obtain two equations in two unknowns, c and h. Interpret both equations. Don't attempt to find the reduced form, it's not analytic. You can try to get the reduced form for $\eta = 0$ and $\eta = 1$ if you like.

   Why do this problem? It's a standard "consumption-leisure" problem, seen often in RBC and New Keynesian models.

3. Now let's solve a "dynamic" problem. The consumer chooses consumption today c_t, consumption tomorrow c_{t+1}, and assets which mature tomorrow a_{t+1}. The problem is:

$$\begin{aligned} \max_{c_t, c_{t+1}, a_{t+1}} \quad & \log c_t + \beta \log c_{t+1} \\ \text{s.t} \quad & c_t + a_{t+1} = y_t \\ & c_{t+1} = y_{t+1} + (1 + r_{t+1}) a_{t+1} \end{aligned}$$

For now, you may ignore uncertainty. The state is (y_t, y_{t+1}, r_{t+1}).

(a) Write down the first-order conditions with respect to c_t, c_{t+1}, a_{t+1}, and the multipliers.

(b) Eliminate the multipliers and show that the Euler equation is: $1/c_t = \beta(1 + r_{t+1})/c_{t+1}$.

Why do this problem? Because it's a standard "two-period consumption problem" and, usefully, is your first taste of Euler equations.

Can you find the reduced form? Hint: combine the budget constraints, think about "lifetime income," and compare this question with your Cobb-Douglas question from above.

Could you do a 3-period problem? What would be different?

2 Bonus Problems

Don't feel compelled to do all of these. They're extras that explore common structures used in macroeconomics.

1. Welcome to your first CES-soup question. Let's do it slowly. The problem is:

$$\begin{aligned} \min_{c_1, c_2} \quad & p_1 c_1 + p_2 c_2 \\ \text{s.t.} \quad & C = \left[c_1^{(\theta-1)/\theta} + c_2^{(\theta-1)/\theta} \right]^{\theta/(\theta-1)} \end{aligned}$$

The idea is that the consumer wishes to minimize the cost of buying the bundle C. We typically call c_1 and c_2 "varieties" of C. Note that there is a lot of symmetry in the problem. I could have thrown some weights in front of c_1 and c_2 in the aggregator, or used different exponents, but let's keep it simple.

(a) Confirm that the consumption aggregator is homogeneous of degree 1 in (c_1, c_2).

(b) Write down first-order conditions with respect to c_1, c_2, and the multiplier.

(c) Write down the "demand function" for c_1 in terms of C, the multiplier, and prices.

(d) Write down the multiplier as a function solely of the two prices. Interpret the multiplier. Call the expression you find P.

Why do this problem? Because it's a baby step towards the next one.

2. A "full-scale" CES-soup problem takes the number of varieties to infinity:

$$\min_{c_i} \quad \int_0^1 p_i c_i di$$
$$\text{s.t.} \quad C = \left[\int_0^1 c_i^{(\theta-1)/\theta} di\right]^{\theta/(\theta-1)}$$

   (a) Take FOCs with respect to c_i and the multiplier.

   (b) Write c_i as a function of C, p_i, and the multiplier.

   (c) Write an expression for the multiplier in terms of p_i.

   This exact problem shows up everywhere in New Keynesian models. It is also commonly seen in models of international trade.

3. Suppose you are a firm producing y_i. You face the demand curve:

$$y_i = \left(\frac{p_i}{P}\right)^{-\theta} Y$$

   You have the production function:

$$y_i = Zh_i + \Phi$$

   where Φ is a fixed cost. You wish to maxmize profits:

$$\max_{p_i, h_i} \frac{p_i y_i}{P} - \frac{W}{P} h_i$$

   (a) Write down the problem as a constrained maximization program with two constraints.

   (b) Find FOCs for y_i, h_i, and the multipliers.

   (c) Does the firm set price equal to marginal cost? If not, what is the markup?

   This is the firm-side counterpart to the problem you did before. The consumer maximizes utility, subject to prices, which generates a demand function. The firm takes that demand function as given, and chooses price to maximize profits.

   You can choose to solve this problem in a number of ways. First, you could maximize with respect to (y_i, p_i, h_i), subject to the demand and production constraints. You could eliminate y_i from the problem and maximize with respect to (p_i, h_i) and one constraint. Or you can reduce everything to a problem in just (p_i) and solve the thing as an unconstrained max problem in (p_i).

   Patience! Some of these problems are less tightly specified than what you're probably used to from undergraduate. Ask questions if you get stuck. Work together on hard problems.

4. IS-LM. Finally, a "macro" model. This is all the IS-LM you're going to get from me. Suppose the IS and LM curves are:

$$Y = [c_0 + c_1(Y - T) - c_2 r] + [d_0 + d_1 Y - d_2 r] + G \tag{IS}$$
$$M = M_0 + kY - hr \tag{LM}$$

I've normalized the price level to 1.

   (a) Solve for (Y, r) in terms of the three policy variables (G, T, M).

   (b) Calculate the entries of the matrix:

$$\begin{bmatrix} dY/dG & dY/dT & dY/dM \\ dr/dG & dr/dT & dr/dM \end{bmatrix}$$

   (c) Market Monetarists: Does dY/dG depend on the monetary policy parameters (M_0, h, k)? What does that tell you about "monetary offset?"

   (d) MMTers: When is $dY/dM = 0$?

   Okay, that's enough of that.

5. Sneak peek at "rational expectations." This is the Lucas monetary surprise model. The structural equations are:

$$m_t + v_t = \pi_t + y_t \tag{AD}$$
$$y_t = \bar{y} + \alpha\left(\pi_t - \pi_t^e\right) \tag{SRAS}$$

The first equation is an "aggregate demand" equation (really, it's just the quantity equation). The second equation is known as a "Lucas aggregate supply curve." Everything is in growth rates. It's not used much anymore, but it's a useful prototype small linear rational expectations model. (See Woodford, "Imperfect Common Knowledge," for a recent revival).

   (a) Write π_t and y_t in terms of $(m_t, v_t, \bar{y}, \pi_t^e)$. This is called a *semi-reduced form,* "semi" because we still have that expected inflation term in there.

   (b) Calculate dy/dm and $d\pi/dm$. Calculate $dy/d\pi^e$ and $d\pi/d\pi^e$.

   (c) Now suppose that $\pi_t^e = E_{t-1}\pi_t$. Solve for the rational expectations equilibrium.

   ...Too fast? Let's take it slowly then.

   From part (a), you should have the inflation equation:

$$\pi_t = \frac{1}{1+\alpha}\left(m_t + v_t - \bar{y} + \alpha\pi_t^e\right).$$

   Now substitute in the expectations term:

$$\pi_t = \frac{1}{1+\alpha}\left(m_t + v_t - \bar{y} + \alpha E_{t-1}\pi_t\right)$$

Next, take expectations of both sides, relative to $t-1$. You get:

$$E_{t-1}\pi_t = E_{t-1}\left[\frac{1}{1+\alpha}\left(m_t + v_t - \bar{y} + \alpha E_{t-1}\pi_t\right)\right]$$

Now we use statistics! The expectations operator is linear, so it passes through addition and multiplication, and we can write:

$$E_{t-1}\pi_t = \frac{1}{1+\alpha}\left(E_{t-1}m_t + E_{t-1}v_t - E_{t-1}\bar{y} + \alpha E_{t-1}E_{t-1}\pi_t\right)$$

This is the exact line where we "invoke rational expectations." Just FYI.

Next, I need to tell you about how those different variables behave. Suppose that:

- The money supply is comprised of a known part $\bar{m}$ and an unknown part, u_t. Then $E_{t-1}m_t = E_{t-1}[\bar{m} + u_t] = \bar{m}$.
- The velocity shock is white noise, so $E_{t-1}v_t = 0$.
- The natural growth rate of output is known, so $E_{t-1}\bar{y} = \bar{y}$.
- What you expect to expect is what you expect, so $E_{t-1}E_{t-1}\pi_t = E_{t-1}\pi_t$.

Make all of those substitutions, and what do we get?

$$E_{t-1}\pi_t = \frac{1}{1+\alpha}\left(\bar{m} - \bar{y} + \alpha E_{t-1}\pi_t\right)$$

which simplifies to:

$$E_{t-1}\pi_t = \bar{m} - \bar{y}$$

Expected inflation is equal to normal nominal growth less normal real growth.

Now we can solve the model. My semi-reduced-forms are:

$$\begin{aligned} y_t &= \frac{1}{1+\alpha}\left(\alpha m_t + \alpha v_t + \bar{y} - \alpha\pi_t^e\right) \\ \pi_t &= \frac{1}{1+\alpha}\left(m_t + v_t - \bar{y} + \alpha\pi^e\right) \end{aligned}$$

And we know now that $\pi_t^e = E_{t-1}\pi_t^e = \bar{m} - \bar{y}$.

Find the reduced form. Verify (really, do it yourself!) that the solution is:

$$y_t = \bar{y} + \frac{\alpha}{1+\alpha}u_t + \frac{\alpha}{1+\alpha}v_t \tag{1}$$

$$\pi_t = \bar{m} - \bar{y} + \frac{1}{1+\alpha}u_t + \frac{1}{1+\alpha}v_t \tag{2}$$

These equations are important enough that I've numbered them.

Final questions: what is $dy/d\bar{m}$? What is dy/du? What is $d\pi/d\bar{m}$? What is $d\pi/du$? What do these results tell you about monetary neutrality? Can you guess why this model is called a "monetary surprise" model?